The cohomology rings of homogeneous spaces

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Conference “Geometry, Groups, Operator Algebras, and Integrability”
Moscow, June 2022

The problem

G compact connected Lie group
$K \subset G$ closed connected subgroup

Theorem (Cartan 1950)

There is an iso of algebras

$$H^*(G/K;\mathbb{R}) \cong \mathrm{Tor}_{H^*(BG)}(\mathbb{R}, H^*(BK))$$

$H^*(BG;\mathbb{R})$ and $H^*(BK;\mathbb{R})$ are polynomial algebras

Problem

Determine $H^*(G/K) = H^*(G/K;\Bbbk)$ for other coefficient rings $\Bbbk$

Assumptions: $H^*(BG;\Bbbk)$, $H^*(BK;\Bbbk)$ are polynomial ($\Bbbk$ a PID). Equivalently, $H^*(G;\mathbb{Z})$, $H^*(K;\mathbb{Z})$ have no torsion elements of order $\operatorname{char}\Bbbk$.

Problem: Singular cochain algebras not (graded) commutative

The solution

Theorem (Baum 1963/68, May '68, Gugenheim–May '74, Munkholm '74, Husemoller–Moore–Stasheff '74, Wolf '77)

There is an iso of $\mathbb{k}$-modules

$$H^*(G/K) \cong \mathrm{Tor}_{H^*(BG)}(\mathbb{k}, H^*(BK))$$

Example (Baum)

$G/K = PU(2) = U(2)/U(1) \approx \mathbb{RP}^3$ and $\mathbb{k} = \mathbb{Z}_2$

The iso is not multiplicative in this case!

Theorem

Assume that 2 is invertible in $\mathbb{k}$. There is an iso of $\mathbb{k}$-algebras

$$H^*(G/K) \cong \mathrm{Tor}_{H^*(BG)}(\mathbb{k}, H^*(BK))$$

natural in the pair (G, K).

Bar construction I

differential graded $\Bbbk$-algebra (**dga**): complex A with associative multiplication $A \otimes A \to A$ such that $d(ab) = (da)b \pm a(db)$

Example: singular cochain algebra $C^*(X)$

Bar construction: assume $A^0 = \Bbbk$, $A^1 = 0$; write $\bar{A} = A^{>0}$

$$\mathbf{B}A = \bigoplus_{n \geq 0} \left(\mathbf{s}^{-1}\bar{A}\right)^{\otimes n} \ni \boldsymbol{a} = [a_1|\ldots|a_n]$$

$$d\boldsymbol{a} = \sum_{k=1}^{n} \pm[\ldots|da_k|\ldots] + \sum_{k=1}^{n-1} \pm[\ldots|a_k a_{k+1}|\ldots]$$

$\mathbf{B}A$ is a dg coalgebra (**dgc**) with diagonal $\Delta\colon \mathbf{B}A \to \mathbf{B}A \otimes \mathbf{B}A$

$$\Delta\boldsymbol{a} = \sum_{k=0}^{n} [a_1|\ldots|a_k] \otimes [a_{k+1}|\ldots|a_n]$$

Bar construction II

One-sided bar construction $\mathbf{B}(\Bbbk, A, M)$ for a dg A-module M:

$\mathbf{B}A \otimes M$ with differential

$$d([a_1|\dots|a_n] \otimes m) = d[a_1|\dots|a_n] \otimes m \pm [a_1|\dots|a_n] \otimes dm \\ \pm [a_1|\dots|a_{n-1}] \otimes a_n m$$

Differential torsion product:

$$\mathrm{Tor}_A(\Bbbk, M) = H^*(\mathbf{B}(\Bbbk, A, M))$$

Theorem (Eilenberg–Moore 1964/66)

For a fibre bundle $F \hookrightarrow E \to B$ with simply connected base B there is an iso of $\Bbbk$-modules

$$H^*(F) \cong \mathrm{Tor}_{C^*(B)}(\Bbbk, C^*(E))$$

Product structure: not easy because $C^*(X)$ not commutative

Homotopy Gerstenhaber algebras

An **hga** is a dga A with operations $E_k\colon A \otimes A^{\otimes k} \to A$ ($E_0 = \mathrm{id}_A$) inducing a dga structure on $\mathbf{B}A$ compatible with the diagonal.

$$\begin{aligned}
[a_1|a_2]\cdot[b_1|b_2] &= [a_1|a_2|b_1|b_2] + [a_1|b_1|a_2|b_2] + [a_1|b_1|b_2|a_2] \\
&\quad + [b_1|a_1|a_2|b_2] + [b_1|a_1|b_2|a_2] + [b_1|b_2|a_1|a_2] \\
&\quad + [E_1(a_1;b_1)|a_2|b_2] + [E_1(a_1;b_1)|b_2|a_2] + [b_1|E_1(a_1;b_2)|a_2] \\
&\quad + [a_1|E_1(a_2;b_1)|b_2] + [a_1|b_1|E_1(a_2;b_2)] + [b_1|a_1|E_1(a_2;b_2)] \\
&\quad + [E_1(a_1;b_1)|E_1(a_2;b_2)] + [E_2(a_1;b_1,b_2)|a_2] + [a_1|E_2(a_2;b_1,b_2)]
\end{aligned}$$

$\cup_1 = \pm E_1$ is a $\cup_1$-product:

$$d(a \cup_1 b) + da \cup_1 b + a \cup_1 db = ab + ba$$
$$(ab) \cup_1 c = (a \cup_1 c)\, b + a \cup_1 (bc)$$

Any commutative dga is an hga with $E_k = 0$ for $k \geq 1$.

Theorem (Baues 1980, Voronov–Gerstenhaber 1995)

$C^*(X)$ is naturally an hga.

A product on the one-sided bar construction

Let $A \to A'$ be an hga map, $\mathbf{B}(\mathbb{k}, A, A') \ni \boldsymbol{a} \otimes a', \boldsymbol{b} \otimes b'$
where $\boldsymbol{a} = [a_1, \ldots, a_m]$, $\boldsymbol{b} = [b_1 | \ldots | b_n] \in \mathbf{B}A$, a', $b' \in A'$

Proposition (Kadeishvili–Saneblidze 2005)

$\mathbf{B}(\mathbb{k}, A, A')$ *is a dga with product*

$$(\boldsymbol{a} \otimes a') \cdot (\boldsymbol{b} \otimes b') = \sum_{l=0}^{n} \boldsymbol{a} \cdot [b_1 | \ldots | b_l] \otimes E_{n-l}(a'; b_{l+1}, \ldots, b_m)\, b'$$
$$= \boldsymbol{ab} \otimes a' b' + \textit{deformation terms}$$

In particular, $\mathbf{B}\big(\mathbb{k}, C^*(B), C^*(E)\big)$ is a dga. The EM iso

$$H^*(F) \cong \mathrm{Tor}_{C^*(B)}\big(\mathbb{k}, C^*(E)\big)$$

then becomes multiplicative.

Eilenberg–Moore for homogeneous spaces

The universal G-bundle $G \hookrightarrow EG \to BG$ gives the bundle

$$G/K \hookrightarrow EG/K = BK \to BG$$

Corollary

There is an isomorphism of algebras

$$H^*(G/K) \cong \mathrm{Tor}_{C^*(BG)}(\mathbb{k}, C^*(BK))$$

Strategy: construct map between one-sided bar constructions underlying $\mathrm{Tor}_{C^*(BG)}(\mathbb{k}, C^*(BK))$ and $\mathrm{Tor}_{H^*(BG)}(\mathbb{k}, H^*(BK))$

One also has:

Proposition (Baum)

$H^*(G/K) \to H^*(G/T)$ *is injective for a maximal torus* $T \subset K$.

A naive approach

Choose generators $x_i \in H^*(BG)$ and $y_j \in H^*(BK)$ as well as cochain representatives for them. This gives chain maps

$$H^*(BG) \to C^*(BG) \qquad \text{and} \qquad H^*(BK) \to C^*(BK) \qquad (*)$$

whence a map

$$\tilde{\Theta} \colon \mathbf{B}(\mathbb{k}, H^*(BG), H^*(BK)) \to \mathbf{B}(\mathbb{k}, C^*(BG), C^*(BK))$$

Problem: $\tilde{\Theta}$ is not a chain map!

Reason 1: the maps $(*)$ are not multiplicative

Reason 2: also not compatible with restriction from BG to BK

This can be overcome via certain kinds of up-to-homotopy structures, namely A_∞ maps and strongly homotopy commutative (shc) algebras. Pioneered by Stasheff–Halperin (1970) and Munkholm (1974).

Set-up of the proof

Theorem

Any "extended hga" (having certain additional operations including a $\cup_2$-product) is canonically an shc algebra.

Applied to $C^*(BG)$ and $C^*(BK)$, this allows to explicitly contruct a (very complicated!) additive quasi-isomorphism

$$\Theta\colon \mathbf{B}\big(\Bbbk, H^*(BG), H^*(BK)\big) \to \mathbf{B}\big(\Bbbk, C^*(BG), C^*(BK)\big)$$

Claim: Θ is multiplicative in cohomology.

By Baum's result, it is enough to look at the composition

$$\mathbf{B}\big(\Bbbk, H^*(BG), H^*(BK)\big) \xrightarrow{\Theta} \mathbf{B}\big(\Bbbk, C^*(BG), C^*(BK)\big) \to \mathbf{B}\big(\Bbbk, C^*(BG), C^*(BT)\big)$$

Hga formality of BT

Theorem (Gugenheim–May)

There is a quasi-iso of dgas

$$f\colon C^*(BT) \to H^*(BT)$$

annihilating all $\cup_1$-products.

Theorem

- *There is a quasi-iso of hgas*

  $$f\colon C^*(BT) \to H^*(BT),$$

  i.e., annihilating all hga operations E_k ($k \geq 1$), and all extended hga operations except for $\cup_2$.
- *Assume that 2 is invertible in $\Bbbk$. Then f can be chosen to additionally annihilate all $\cup_2$-products of cocycles.*

Completing the proof

Corollary: There is a quasi-iso of dgas

$$\Psi\colon \mathbf{B}\big(\mathbb{k}, C^*(BG), C^*(BT)\big) \to \mathbf{B}\big(\mathbb{k}, C^*(BG), H^*(BT)\big)$$

Advantage: the product on the right is componentwise

$$(\boldsymbol{a} \otimes a) \cdot (\boldsymbol{b} \otimes b) = \pm \boldsymbol{ab} \otimes ab$$

Hence enough to look at the composition

$$\mathbf{B}\big(\mathbb{k}, H^*(BG), H^*(BK)\big) \xrightarrow{\Theta} \mathbf{B}\big(\mathbb{k}, C^*(BG), C^*(BK)\big) \\ \to \mathbf{B}\big(\mathbb{k}, C^*(BG), C^*(BT)\big) \xrightarrow{\Psi} \mathbf{B}\big(\mathbb{k}, C^*(BG), H^*(BT)\big)$$

Now one goes through the construction of Θ and identifies all deformation terms that end up in the kernel of Ψ. The remaining terms are multiplicative up to homotopy if 2 is invertible in $\mathbb{k}$.

References

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